DOCUMENTS DE TREBALL

DE LA FACULTAT D'ECONOMIA I EMPRESA

Col.lecció d'Economia

E09/214

A note on the coincidence between Stackelberg and Nash equilibria in a differential game between government and firms

Jesús Marín-Solano

Jorge Navas

Adreça correspondència:
Departamento de Matemática Económica, Financiera y Actuarial
Facultat d'Economia i Empresa
Universitat de Barcelona
Av. Diagonal 690
08034 Barcelona

Abstract

In Navas and Marín-Solano (2008) the coincidence between Nash and Stackelberg equilibria for a modified version of the differential game model first proposed by Lancaster (1973) was proved. However, important restrictions on the value of the parameters of the model were included, in order to obtain an interior solution. In this paper we extend the previous result, in the limit when the discount rate is equal to zero, by eliminating the restrictions and taking into account corner solutions.

Resum

En Navas i Marín-Solano es va demostrar la coincidència entre els equilibris de Nash i de Stackelberg per a una versió modificada del joc diferencial proposat por Lancaster (1973). Amb l'objectiu d'obtenir una solució interior, es van imposar restriccions importants sobre el valors dels paràmetres del model. En aquest treball estenem aquest resultat, en el límit en que la taxa de descompte és igual a zero, eliminant les restriccions i considerant totes les solucions possibles.

JEL classification: C73; H21; H32

Keywords: Nash/Stackelberg equilibria; optimal profit taxation; time consistency

1 Introduction

In this paper we study the coincidence between Nash and Stackelberg equilibria for an extension of the Lancaster capitalism model (Lancaster (1973), Pohjola (1983), de Zeew (1992)), which was later on adapted to analyze an optimal taxation policy problem in Gradus (1989).

We consider the following differential game. There are two players. The control (decision) variable of Player 1 is $u(t) \in [b, c]$, with $0 < b < c < 1$. The control variable of Player 2 is $v(t) \in [0, 1]$. The state of the system is described by the state variable $K(t)$ (the stock of capital), which evolves according to

$$\dot{K} = aK(1-u)v \ , \qquad K(0) = K_0 > 0 \ . \tag{1}$$

The objective of the two agents are, for Player 1,

$$\max_u \left\{ J_1 = \int_0^T aKudt + \alpha K(T) \right\} \tag{2}$$

and, for Player 2,

$$\max_v \left\{ J_2 = \int_0^T aK(1-u)(1-v)\,dt + (1-\alpha)K(T) \right\} \ . \tag{3}$$

In the Gradus optimal taxation model, Players 1 and 2 are government and firms, u represents the taxes on the profits of the firms, a is the the ratio profit-stock of capital, and α can be seen as a net wealth tax rate applied at the end of the time horizon. In the capitalism model (Lancaster (1973)), Players 1 and 2 represent workers and capitalists, u is the share of consumption in total output of workers, and α plays a role similar to the one by u at the end of the planning horizon. In both models, v is the investment rate of capitalists or firms.

The model falls within the class of trilinear differential games (see Clemhout and Wan (1974)), which have the property that the open-loop and feedback Nash equilibria coincide. Moreover, as it is argued in Basar *et al* (1985), the feedback Nash and Stackelberg solutions with Player 1 as the leader also coincide for this class of differential games. In Navas and Marín-Solano (2008) it was proved that, for certain values of the parameters of the model (those guaranteeing the existence of interior solutions), Player 1 can choose the parameter α in such a way that the open-loop Stackelberg equilibrium (with Player 1 as the leader) coincides with the Nash equilibrium, so the open-loop Stackelberg equilibrium becomes time consistent. Moreover, this value of α maximizes the payments for the two players when the other time consistent nonooperative equilibria are considered. In this paper we generalize this result (in the limit when the discount rate is equal to 0) by relaxing the conditions of the model in the sense that corner solutions are permitted.

2 Nash and Stackelberg equilibrium: Player 1 as the leader

Nash equilibrium. For the derivation of the open-loop Nash equilibrium (and hence the other equilibrium solutions) for the differential game defined by (1-3) we consider the Hamiltonian functions $H_1 = aKu + p_1 aK(1-u)v$ and

$$H_2 = aK(1-u)(1-v) + p_2 aK\,(1-u)\,v\ , \tag{4}$$

where p_1 and p_2 are the corresponding co-state variables. The maximum of H_1 in u is achieved when $u^* = b$ if $p_1 v > 1$, and $u^* = c$ if $p_1 v < 1$. The maximum in v of H_2 is achieved when $v^* = 0$ if $p_2 < 1$, and $v^* = 1$ if $p_2 > 1$. In addition, the following set of differential equations must be satisfied:

$$\dot{p}_1 = -\frac{\partial H_1}{\partial K} = -a\,[u + p_1(1-u)v]\ , \quad p_1(T) = \alpha\ , \tag{5}$$

$$\dot{p}_2 = -\frac{\partial H_2}{\partial K} = -a\,[1 + (p_2 - 1)\,v]\,(1-u)\ , \quad p_2(T) = (1-\alpha)\ . \tag{6}$$

Note that $p_1(t)$ and $p_2(t)$ are continuous and strictly decreasing functions. We assume that $p_2(0) > 1$ (if $p_2(0) \leq 1$, the dynamics is obtained as a truncated analysis of the previous case). From (6), $p_2(t_2^N) = 1$ if, and only if,

$$t_2^N = T - \frac{\alpha}{a(1-c)}\ .$$

For every $t \in (t_2^N, T]$, $u^*(t) = c$, $v^*(t) = 0$, and $K(t) = K_s$ (constant).

Let $t \in [0, t_2^N)$. Then $v^*(t) = 1$, and $u^*(t) = b$ if $p_1(t) > 1$, $u^*(t) = c$ if $p_1(t) < 1$. Since $p_1(t)$ is strictly decreasing, there exists at most a moment $t_1^N \in [0, t_2^N)$ such that $p_1(t_1^N) = 1$. If $p_2(0) > 1 \geq p_1(0)$, $u^* = c$ for every $t \in [0, T]$. For the general case when $p_1(0), p_2(0) > 1$ we consider two cases:

1. If $\alpha > 1 - c$, $p_1(t_2^N) \geq 1$, $u^*(t) = b$ if $t \in [0, t_2^N)$, and $u^*(t) = c$ if $t \in [t_2^N, T]$.

2. If $\alpha < 1 - c$, $p_1(t_2^N) < 1$ and there exists $t_1^N \in (0, t_2^N)$ such that $p_1(t_1^N) = 1$. For every $t \in [0, t_1^N)$, $u^*(t) = b$; and for every $t \in (t_1^N, T]$, $u^*(t) = c$. Since $p_1(t_2^N) - 1 = (\alpha + c - 1)/(1-c)$, from (5) the switching point t_1^N is given by

$$t_1^N = t_2^N + \frac{1}{a(1-c)} \ln\,[c + \alpha]\ .$$

Open-loop Stackelberg equilibrium: Player 1 as the leader. For the resolution of the open-loop Stackelberg equilibrium, we follow an approach similar to the one in de Zeeuw (1992). Let us analyze the problem of the follower, whose Hamiltonian function is given by (4). The dynamics of the state and costate variables K and p_2 is described by (1) and (6). We assume that $p_2(0) > 1$. Since $p_2(T) < 1$ and $p_2(t)$ is a strictly decreasing continuous function, there exists a unique moment $t_2^S \in [0,T]$ such that $p_2(t^S) = 1$. For $t < t_2^S$, $v^*(t) = 1$, and for $t > t_2^S$, $v^*(t) = 0$. By integrating (6) along the interval $t \in (t_2^S, T]$ we obtain

$$\int_{t_2^S}^{T} u \, \mathrm{d}t = T - t_2^S - \frac{\alpha}{a} \, . \tag{7}$$

Now, in the problem for the leader, since the switching point t_2^S defines the change of the investment strategy for the follower, from (2) and (7) we obtain

$$J_1^S = \int_0^{t_2^S} aKudt + \int_{t_2^S}^{T} aKudt + \alpha K(T) = \int_0^{t_2^S} aKudt + aK\left(t_2^S\right)\left[T - t_2^S\right] \, . \tag{8}$$

Player 1 will choose the control variable $u(t)$, $t \in [0, t_2^S)$ ($t_2^S \leq T$), which maximizes (8), with the constraints (7) and $\dot{K} = aK(1-u)$, $K(0) = K_0 > 0$. The terminal time t_2^S is free. Since $u \in [b,c]$, t_2^S is bounded, $t_2^S \in [T - \alpha/(a(1-c)), T - \alpha/(a(1-b))]$, to be able to meet (7). We have to study two cases.

Case 1: Interior solution. If $t_2^S \in (T - \alpha/(a(1-c)), T - \alpha/(a(1-b)))$, the Hamiltonian function is $\tilde{H}_1 = aKu + \tilde{p}_1 aK(1-u)$, where $\tilde{p}_1$ is the co-state variable of Player 1. The maximum in u of $\tilde{H}_1$ is b if $\tilde{p}_1 > 1$, and c if $\tilde{p}_1 < 1$. Denoting the final function by $F_1 = aK(t^S)(T - t^S)$ then, for every $t < t_2^S$,

$$\begin{aligned} \dot{\tilde{p}}_1 &= -\frac{\partial \tilde{H}_1}{\partial K} = -\left[au + \tilde{p}_1 a(1-u)\right] \, , \; \tilde{p}_1(t_2^S) = \frac{\partial F_1}{\partial K(t_2^S)} = a(T - t_2^S) \, , && (9) \\ 0 &= \tilde{H}_1(t^S) + \frac{\partial F_1}{\partial t^S} \, . && (10) \end{aligned}$$

Using (9-10), a simple calculation shows that the (unique) switching point is given by $t_2^S = T - 1/a$, and does not depend on the parameter α. The following table resumes the results obtained in the case of interior solution:

$u(t) = b$	$v(t) = 1$	$t \in [0, T - 1/a)$
$\int_{T-1/a}^{T} u(t)\, dt = (1-\alpha)/a$	$v(t) = 0$	$t \in (T - 1/a, T]$

(11)

Note that, from (7), we have $\int_{t_2^S}^{T} u \, dt = \frac{1-\alpha}{a}$. For instance, if we take a constant control variable $u = \bar{u}$ from t_2^S to T, we obtain $\bar{u} = 1 - \alpha$. Hence, if $1 - \alpha \notin [b,c]$, the restriction (7) cannot be met. Therefore, the solution is interior if, and only if, $b \leq 1 - \alpha \leq c$. Otherwise, we obtain a corner solution.

Case 2: Corner solutions. If $\alpha > 1 - b$ or $\alpha < 1 - c$, the free final time t_2^S takes values in the boundary of the interval $[T - \alpha/(a(1-c)), T - \alpha/(a(1-b))]$ and we have to replace (10) by

$$\begin{aligned} \tilde{H}_1(t_2^S) + \frac{\partial F_1}{\partial t_2^S} &\geq 0 \quad \text{if} \quad t_2^S = T - \frac{\alpha}{a(1-b)} \, , \quad \text{and} \\ \tilde{H}_1(t_2^S) + \frac{\partial F_1}{\partial t_2^S} &\leq 0 \quad \text{if} \quad t_2^S = T - \frac{\alpha}{a(1-c)} \, . \end{aligned}$$

Without loss of generality, let us assume a constant value for the control variable $u(t)$ throuhout the interval $[t_2^S, T]$. From (7) we obtain $t_2^S = T - \alpha/(a(1-u(t_2^S)))$, hence $\tilde{H}_1(t_2^S) + \dfrac{\partial F_1}{\partial t_2^S} = aK(t_2^S)(\alpha - 1 + u(t_2^S))$.

1. If $\alpha > 1-b$, then $\tilde{H}_1(t_2^S) + \partial F_1/\partial t_2^S > 0$, so $t_2^S = T - \alpha/(a(1-b))$. Since $\tilde{p}_1(t_2^S) = \alpha/(1-b) > 1$, then $u^*(t) = b$, for every $t \in [0, T]$. The we have:

$u(t) = b$	$v(t) = 1$	$t \in [0, T - \alpha/(a(1-b)))$
$u(t) = b$	$v(t) = 0$	$t \in [T - \alpha/(a(1-b)), T)$

(12)

2. If $\alpha < 1-c$, then $\tilde{H}_1(t_2^S) + \partial F_1/\partial t_2^S < 0$, so $t_2^S = T - \alpha/(a(1-c))$ and $u^*(t) = c$, for $t \geq t_2^S$. Now $\tilde{p}_1(t_2^S) = \alpha/(1-c) < 1$, hence there exists t_1^S such that $u^*(t) = b$ if $t < t_1^S$, and $u^*(t) = c$ if $t > t_1^S$. By solving (9) we obtain that

$$t_1^S = T - \frac{1}{a(1-c)} \left[\alpha - \ln(\alpha + c)\right] \ .$$

is the unique moment such that $\tilde{p}_1(t_1^S) = 1$. Then we have

$u(t) = b$	$v(t) = 1$	$t \in [0, T - [\alpha - \ln(\alpha + c)]/(a(1-c))$
$u(t) = c$	$v(t) = 1$	$t \in [T - [\alpha - \ln(\alpha + c)]/(a(1-c)), T - \alpha/(a(1-c)))$
$u(t) = c$	$v(t) = 0$	$t \in (T - \alpha/(a(1-c)), T]$

(13)

3 Main results

Let us analyze the consequences of the existence of the parameter α.

Proposition 1 *The maximum in α of the payment for Player 1, $(J_1^S)^*$, in the open-loop Stackelberg equilibria, is achieved for every $\alpha \in [1-c, 1-b]$.*

Proof: If the solution is interior, using (1), (8) and (11) it is easy to show that J_1^S does not depend on the value of the parameter α, i.e., $\partial(J_1^S)^*/\partial\alpha = 0$.

If $\alpha > 1-b$, we have the corner solution (12). From (8) we obtain

$$(J_1^S)^* = ab \int_0^{t_2^S} K\, dt + \frac{\alpha}{1-b} K(t_2^S) \ .$$

Hence, by integrating (1) and substituting in the previous expression we obtain

$$\frac{\partial (J_1^S)^*}{\partial \alpha} = \frac{1-b-\alpha}{1-b} K(t_2^S) < 0 \ .$$

Finally, if $\alpha < 1-c$, a similar calculation for the corner solution described in (13) shows that, in this case, $\dfrac{\partial (J_1^S)^*}{\partial \alpha} > 0$. Then the result follows. □

With respect to the second player, similar calculations shows that $\frac{\partial (J_2^S)^*}{\partial \alpha} = 0$ if $\alpha \in (1-c, 1-b)$, and $\frac{\partial (J_2^S)^*}{\partial \alpha} < 0$ if $\alpha > 1-b$ or $\alpha < 1-c$. Therefore, Player 2 maximizes its payments when α takes the minimum value.

Let us compare the switching points in the Nash and Stackelberg equilibria, which characterize completely the solutions, so a coincidence in the switching points implies equal payments for both players. Simple calculations illustrate that, if the solution is interior, the switching points in the Nash and Stackelberg equilibria coincide if, and only if, $\alpha = 1-c$. In the case of corner solutions, if $\alpha < 1-c$, the switching points always coincide. On the contrary, if $\alpha > 1-b$, the corresponding switching points do not coincide. As a corollary we obtain:

Proposition 2 *If the parameter α takes the value $\alpha^* = 1-c$, then the open-loop and feedback Nash equilibria, and the open-loop and feedback Stackelberg equilibria with Player 1 as the leader coincide.*

Finally, we prove that the open-loop Stackelberg equilibria (with Player 1 as the leader) give greater payments to both agents than the open-loop Nash equilibrium.

Proposition 3 *Both players prefer the open-loop Stackelberg equilibria to the open-loop Nash equilibrium, i.e., $(J_1^S)^* \geq (J_1^N)^*$ and $(J_2^S)^* \geq (J_2^N)^*$.*

Proof: The result for Player 1 is straightforward (he is the leader). With respect to Player 2, if $\alpha \leq 1-c$, the switching point coincide, so $(J_2^S)^* = (J_2^N)^*$. For $\alpha \geq 1-c$, note that $J_2^N = K(t_2^N)$. If $\alpha > 1-b$, several calculations show that

$$(J_2^N)^* = K_0 e^{1(1-b)T-\alpha\left(\frac{1-b}{1-c}\right)} < K_0 e^{1(1-b)T-\alpha} = (J_2^S)^* \ .$$

Finally, if $\alpha \in (1-c, 1-b)$, for the unique switching point in the Nash equilibrium it is clear that $\frac{\partial K(t_2^N)}{\partial t_2^N} > 0$, and also $\frac{\partial t_2^N}{\partial \alpha} < 0$. Therefore, $\frac{\partial (J_2^N)^*}{\partial \alpha} < 0$ and the result follows ($\frac{\partial (J_2^S)^*}{\partial \alpha} = 0$ in the interior solutions). □

From Propositions 1, 2 and 3, if $\alpha^* = 1-c$ the open-loop Stackelberg equilibrium is time consistent, thus providing an answer to the main criticism of this solution concept. Moreover, since the payments for Player 1 (and also for Player 2 if $\alpha \geq 1-c$) are maximum in the open-loop Stackelberg equilibria for $\alpha \in [1-c, 1-b]$, $\alpha^* = 1-c$ can be seen as the maximizer of payments for Player 1 when the open-loop/feedback Nash equilibrium, or the feedback Stackelberg solution with Player 1 as the leader, are considered. In the differential game model between government and firms, α can be interpreted as a net wealth tax, which is fixed by the government (Player 1). In the model of capitalism, it seems more natural to assume that α is choosed as a consequence of a bargaining process between the capitalist (Player 2) and workers (Player 1).

References

[1] Basar, T., Haurie, A. and G. Ricci, 1985, On the dominance of capitalist leadership in a 'feedback-Stackelberg' solution of a differential game model of capitalism, Journal of Economic Dynamics and Control 9, 101-125.

[2] Clemhout, S. and H.Y. Wan Jr., 1974, A Class of Trilinear Differential Games, Journal of Optimization Theory and Applications 14, 419-424.

[3] Gradus, R., 1989, A Differential Game Between Government and Firms: A Non-Cooperative Approach, Journal of Economics 50, 237-256.

[4] Lancaster, K., 1973, The dynamic inefficiency of capitalism, Journal of Political Economy 81, 1092-1109.

[5] Navas, J. and J. Marín-Solano, 2008, Interactions between government and firms: a differential game approach, Annals of Operations Research 158, 47-61.

[6] Pohjola, M., 1983, Nash and Stackelberg solutions in a differential game model of capitalism, Journal of Economic Dynamics and Control 6, 173-186.

[7] de Zeeuw, A., 1992, Note on 'Nash and Stackelberg solutions in a differential game model of capitalism, Journal of Economic Dynamics and Control 16, 139-145.